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Oddball determinants

J.S.Dowker*

*Department of Theoretical Physics,
The University of Manchester, Manchester, England.*

Abstract

A simplified direct method is described for obtaining massless scalar functional determinants on the Euclidean ball. The case of odd dimensions is explicitly discussed.

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*Dowker@a3.ph.man.ac.uk

1. Introduction

In an earlier work, [1], a method of evaluating functional determinants of the Laplacian, D, on the even Euclidean ball was presented. In practice, the technique was over-elaborate. It used the Watson-Kober summation result, an exact formula involving Bessel functions, and thereby, since only an asymptotic limit was required, unrequired information was introduced, necessitating algebraic contortions in order to reach the final goal. While some of these manipulations turned up interesting identities and summations, they were strictly unnecessary and somewhat of a luxury. The object of the present, brief work is to outline an improved method that emphasises the asymptotic behaviour more immediately. The technique retains an element of the Watson-Kober approach and generalises that of Moss [2] employed by him *en passant* while evaluating heat-kernel coefficients.

2. Basic formulae

Only a few of the starting formulae will be given here. Since our previous work concerned even balls we concentrate, in the particular, on odd d-balls. Hence the title of this paper.

The basic equation is (the $\approx$ sign means equal to the mass-independent part of the large m^2 asymptotic limit)

$$\zeta'(0) = -\ln \det D \approx \lim_{m \to \infty} \ln \Delta(-m^2), \tag{1}$$

where Δ is the Weierstrass product,

$$\Delta(-m^2) = \prod_{p,\alpha_p} (1 + \frac{m^2}{\alpha_p^2}) \exp \sum_{k=1}^{[d/2]} \frac{1}{k} \left(\frac{-m^2}{\alpha_p^2}\right)^k, \tag{2}$$

α_p^2 being the eigenvalues of the relevant equation obtained by setting combinations of Bessel functions of order p to zero. For simplicity the equations are developed for Dirichlet conditions and then the Mittag-Leffler theorem implies

$$\Delta(-m^2) = \prod_p \left(p! 2^p m^{-p} I_p(m^2)\right) \exp \sum_{k=1}^{[d/2]} \frac{1}{k} \left(\frac{-m^2}{\alpha_p^2}\right)^k \tag{3}$$

in terms of the modified Bessel function I_p.

The degeneracy, $N_p^{(d)}$, which is implied by the product in (3), is, for odd balls, an odd polynomial in p. Hence it is enough to consider the quantity, [1],

$$A_\nu = \ln \Delta(-m^2) \sim \sum_{p=1/2}^{\infty} p^{2\nu+1} \left[p \ln \frac{2p}{p+\epsilon} + \epsilon - p - \frac{1}{2} \ln \frac{\epsilon}{p} + \sum_{n=1}^{\infty} \frac{T_n(t)}{\epsilon^n} \right.$$
$$\left. - \operatorname{Ray}(m,p) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} dt \right], \tag{4}$$

where $\operatorname{Ray}(m,p)$ are the 'Rayleigh' terms (*i.e.* the sums of inverse even powers of the eigenvalues). Olver's asymptotic form of I_p has been employed and an integral representation for $\ln p!$ substituted. The abbreviation $t = p/\epsilon$ with $\epsilon^2 = p^2 + m^2$ is frequently used.

The central idea of simply discarding any m-dependent contributions to (4) is the same as before, [1], and many of our arguments are unchanged. Thus we immediately throw away the Rayleigh terms. Also the discussion of the T_n polynomial contribution and the integral is identical. The major improvement is an accelerated treatment of the remaining terms. To this end we set up the relevant asymptotic formulae.

3. Asymptotic formulae

Some standard results were given in our our earlier work and are repeated here but in slightly generalised forms. The essential results are

$$\sum_{p=1}^{\infty} \frac{p^{2\nu+1}}{\epsilon^{2s}} \sim \sum_{h=0}^{\infty} (1-h) \ldots (\nu - h)(m^2)^{1-h-s+\nu} \frac{\Gamma(s-1+h-\nu)}{2\Gamma(s)} \frac{(-1)^h B_{2h}}{h!} \tag{5}$$

and

$$\sum_{p=1}^{\infty} \frac{p^{2\nu}}{\epsilon^{2s}} \sim \frac{\Gamma(\nu+1/2)\Gamma(s-\nu-1/2)}{2\Gamma(s)} m^{1-2s+2\nu}, \quad \nu > 0. \tag{6}$$

From these formulae it is easy to derive

$$\sum_{p=1/2}^{\infty} \frac{p^{2\nu+1}}{\epsilon^{2s}}$$
$$\sim \sum_{h=0}^{\infty} (1-h) \ldots (\nu - h)(2^{1-2h} - 1)(m^2)^{1-h-s+\nu} \frac{\Gamma(s-1+h-\nu)}{2\Gamma(s)} \frac{(-1)^h B_{2h}}{h!} \tag{7}$$

and

$$\sum_{p=1/2}^{\infty} \frac{p^{2\nu}}{\epsilon^{2s}} \sim \frac{\Gamma(\nu+1/2)\Gamma(s-\nu-1/2)}{2\Gamma(s)} m^{1-2s+2\nu}, \quad \nu > 0, \tag{8}$$

which are more suitable here.

4. Evaluation of the determinant

Now we look at the first three terms in the bracket in (4). Since the entire expression (4) converges as a summation, it is possible to render any part finite by systematic addition and subtraction of appropriate terms. This is done by the removal of enough of the Taylor series to regularise the summation, a process denoted by a star. For example,

$$
\sum_{p=1/2}^{\infty} p^{2\nu+1} \ln\left(\frac{\epsilon}{p}\right) \rightarrow \sum_{p=1/2}^{*} p^{2\nu+1} \ln\left(\frac{\epsilon}{p}\right)
$$
$$
\equiv \sum_{p=1/2}^{\infty} p^{2\nu+1} \left(\ln\left(\frac{\epsilon}{p}\right) - \frac{1}{2} \sum_{k=1}^{\nu+1} (-1)^{k+1} \frac{m^{2k}}{p^{2k}} \right). \tag{9}
$$

Consider now the same procedure applied to the quantities in (7) and (8)

$$
\sum_{p}^{*} \frac{p^N}{\epsilon^{2s}} = \sum_{p}^{\infty} p^N \left[\frac{1}{\epsilon^{2s}} - \frac{1}{p^{2s}} - \sum_{k=1}^{M} \binom{-s}{k} \frac{m^{2k}}{p^{2k+2s}} \right] \tag{10}
$$

whose derivative at $s = 0$ gives minus twice (9) when $N = 2\nu + 1$ and $M = \nu + 1$.

Using (7) on the first term on the right-hand side in (10), performing the summations on the others to give Riemann ζ–functions and then differentiating, one obtains the asymptotic limit of (9). The only term independent of m^2 is clearly

$$
\zeta_R'(-2\nu - 1, 1/2) = (2^{-2\nu-1} - 1)\zeta_R'(-2\nu - 1) + 2^{-2\nu-1}\zeta_R(-2\nu - 1). \tag{11}
$$

We next turn to the term which caused most problems in our earlier calculation,

$$
\sum_{p=1/2}^{\infty} p^{2\nu+2} \ln\frac{2p}{p+\epsilon} = \sum_{p=1/2}^{\infty} p^{2\nu+2} \left[\ln\frac{2p}{\epsilon} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{p^k}{\epsilon^k} \right]
$$
$$
= \sum_{p=1/2}^{\infty} p^{2\nu+2} \left[\ln\frac{p}{\epsilon} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{p^k}{\epsilon^k} - 1 \right) \right], \tag{12}
$$

after an expansion in p/ϵ.

The regularised version of this can be formally rearranged,

$$
\sum_{p=1/2}^{*} p^{2\nu+2} \ln\frac{2p}{p+\epsilon} = \sum_{p=1/2}^{*} p^{2\nu+2} \ln\frac{p}{\epsilon} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{p=1/2}^{*} \frac{p^{2\nu+2+k}}{\epsilon^k} \tag{13}
$$

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so that the asymptotic formulae can again be used. The logarithm has effectively already been considered and (10) can be applied to the last summation setting $s = k/2$ and $N = 2\nu + 2 + k$. The mass-independent term, for all k (even and odd), is easily seen to equal $\zeta_R(-2\nu - 2, 1/2)$ which is zero, whence the total is

$$-\zeta'_R(-2\nu - 2, 1/2) = -(2^{-2\nu-2} - 1)\,\zeta'_R(-2\nu - 2). \tag{14}$$

Incidentally, the corresponding calculation for even balls has the factor $p^{2\nu+1}$ in (12) and a summation over integers. It rapidly gives the answer found in [1],

$$-\zeta'_R(-2\nu - 1) - \ln 2\,\zeta_R(-2\nu - 1),$$

without the complicated manipulations of MacDonald functions that occupied us so excessively there.

The asymptotic behaviour of the remaining term in (4) follows from (10) with $N = 2\nu + 1$ and $s \to -1/2$ and is

$$\sum_{p}^{\infty} p^{2\nu+1}\big(\epsilon - p\big) \to \sum_{p}^{*} p^{2\nu+1}(\epsilon - p) = \sum_{p}^{\infty} p^{2\nu+1}\left(\epsilon - p - \frac{1}{2}\frac{m^2}{p}\right),$$

giving a mass-independent part of $-\zeta_R(-2\nu - 2, 1/2)$, which vanishes.

As remarked, the general discussion of the remaining terms in (4) is exactly that of our previous work. This time there is no escape from the algebra but since the details are a little different from those earlier ones, and slightly easier, some elaboration is not out of place here.

As explained in [1], by adding and subtracting terms, and discarding obviously mass-dependent convergent contributions (in the limit) from the T_n polynomials, one arrives at

$$\sum_{p=1/2}^{\infty} p^{2\nu+1}\Bigg[\sum_{n=1}^{2\nu+2}\left(\frac{T_n(t)}{\epsilon^n} - \frac{T_n(1)}{p^n}\right)$$
$$+ \int_0^\infty \left(\frac{1}{2} - \frac{1}{\tau} + \sum_{k=1}^{\nu+1}(-1)^k B_{2k}\frac{\tau^{2k-1}}{(2k)!} + \frac{1}{e^\tau - 1}\right)\frac{e^{-\tau p}}{\tau}\,d\tau\Bigg] \tag{15}$$

for these remaining terms, the parts of which are now considered. First

$$\sum_{p=1}^{\infty} p^{2\nu+2}\sum_{n=1}^{2\nu+1}\left(\frac{T_n(t)}{\epsilon^n} - \frac{T_n(1)}{p^n}\right). \tag{16}$$

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Repeating our previous analysis we set

$$T_n(t) = T_n(1) + T'_n(t)$$

and get for (16)

$$\sum_{n=1,3,\ldots}^{2\nu+1} \sum_{p=1/2}^{\infty} T_n(1) p^{2\nu+1} \left(\frac{1}{\epsilon^n} - \frac{1}{p^n} \right) + \sum_{n=1}^{2\nu+2} \sum_{p=1/2}^{\infty} \frac{p^{2\nu+1}}{\epsilon^n} T'_n(t) \qquad (17)$$

noting that $T_n(1)$ vanishes for n even.

We apply (10) with $N = 2\nu + 1$ and $s = n/2$ to the first summation. Doing the summations in (10), we note that none of the Riemann ζ–function arguments can equal 1, because $2s$ is odd. For the same reason, there is no mass dependence in the asymptotic limit, (7), of the first term on the right-hand side of (10). Since the last summation in (10) can be discarded as mass-dependent, this just leaves the contribution of the second term, which is

$$- \sum_{n=1,3,\ldots}^{2\nu+1} T_n(1)\, \zeta_R(n - 2\nu - 1, 1/2)$$

and vanishes because n is odd.

In the second summation in (17) only the convergent term $n = 2\nu + 2$ yields a mass-independent part to the asymptotic limit according to (7) and manipulations similar to those outlined in our earlier work produce the value

$$\int_0^1 t^{2\nu+1} T''_{2\nu+2}(t)\, dt \qquad (18)$$

for this constant contribution where $T'_n(t) = (1 - t^2) T''_n(t)$.

Turning to the integral in (15), the sum over p is performed and a regularisation introduced to give

$$\lim_{s \to 0} \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{\tau} + \sum_{k=1}^{\nu+1} (-1)^k B_{2k} \frac{\tau^{2k-1}}{(2k)!} + \frac{1}{e^\tau - 1} \right) \tau^{s-1} (-1)^{2\nu+1} \frac{d^{2\nu+1}}{d\tau^{2\nu+1}} \frac{e^{\tau/2}}{e^\tau - 1}\, d\tau. \qquad (19)$$

The power terms in (19) are dealt with using the formula

$$(-1)^j \int_0^{\infty} \tau^\mu \tau^{s-1} \frac{d^j}{d\tau^j} \frac{e^{\tau/2}}{e^\tau - 1}\, d\tau = \Gamma(\mu + s) \zeta_R(\mu - j + s, 1/2), \qquad (20)$$

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while the remaining part,

$$(-1)^{2\nu+1} \lim_{s \to 0} \int_0^\infty \frac{1}{e^\tau - 1} \tau^{s-1} \frac{d^{2\nu+1}}{d\tau^{2\nu+1}} \frac{e^{\tau/2}}{e^\tau - 1} \, d\tau, \tag{21}$$

is best treated by writing

$$(-1)^j \frac{d^j}{d\tau^j} \frac{e^{\tau/2}}{e^\tau - 1} = \sum_{l=1}^{j+1} D_l^{(j)} \frac{e^{\tau/2}}{(e^\tau - 1)^l}$$

with the recursion

$$\begin{aligned} D_l^{(j)} &= (l - 1/2) D_l^{(j-1)} + (l-1) D_{l-1}^{(j-1)}, \quad 2 \le l \le j, \\ D_{j+1}^{(j)} &= j! \\ D_1^{(j)} &= \frac{1}{2^j}. \end{aligned} \tag{22}$$

A special value is $D_j^{(j)} = jj!/2$.

Expression (21) can be rewritten as

$$\lim_{s \to 0} \sum_{l=1}^{2\nu+2} D_l^{(2\nu+1)} \Gamma(s) \, \zeta_{l+1}(s, l+1/2) \tag{23}$$

in terms of the Barnes ζ–function,

$$\begin{aligned} \zeta_r(s, a) &= \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{z(r-a)} (-z)^{s-1}}{(e^z - 1)^r} \, dz \\ &= \sum_{n=0}^\infty \binom{n+r-1}{r-1} \frac{1}{(a+n)^s}, \quad \mathrm{Re}\, s > r. \end{aligned} \tag{24}$$

In the present case $r = l + 1$, $a = l + 1/2$ and the lower limit can be adjusted so that

$$\zeta_{l+1}(s, l+1/2) = \sum_{n=0}^\infty \binom{n}{l} \frac{1}{(n+1/2)^s}.$$

As usual the numerator is expanded in powers of $(n+1/2)$ using Stirling numbers,

$$(n+a)(n+a-1)\dots(n+a-b+1) = \sum_{k=0}^b T^{(k)}(a, b) \, (n+1/2)^k,$$

to give the Barnes function as a series of standard Riemann-Hurwitz ζ–functions,

$$\zeta_{l+1}(s, l+1/2) = \frac{1}{l!} \sum_{k=0}^l T^{(k)}(0, l) \, \zeta_R(s - k, 1/2), \tag{25}$$

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which are the natural quantities in the odd case.

Putting the two parts of the integral together gives

$$\lim_{s \to 0} \left(\frac{1}{2} \Gamma(s) \zeta_R(s - 2\nu - 1, 1/2) - \Gamma(s - 1) \zeta_R(s - 2\nu - 2, 1/2) \right.$$

$$- \sum_{k=1}^{\nu+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \Gamma(s + 2k - 1) \zeta_R(s + 2k - 2\nu - 2, 1/2) \qquad (26)$$

$$\left. + \Gamma(s) \sum_{l=1}^{2\nu+2} \sum_{k=0}^{l} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!} \zeta_R(s - k, 1/2) \right).$$

The cancellation of the individual divergences is a check of the analysis and implies the identity

$$\frac{1}{2} \zeta_R(-2\nu - 1, 1/2) + \sum_{l=1}^{2\nu+2} \sum_{k=0}^{l} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!} \zeta_R(-k, 1/2) = 0, \qquad (27)$$

where k must be odd.

The finite remainder in (26) is

$$\frac{1}{2} \zeta_R'(-2\nu - 1, 1/2) - \frac{1}{2} \gamma \zeta_R(-2\nu - 1) + \zeta_R'(-2\nu - 2, 1/2)$$

$$+ \sum_{l=1}^{2\nu+2} \sum_{k=0}^{l} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!} \left(\zeta_R'(-k, 1/2) - \gamma \zeta_R(-k, 1/2) \right)$$

which can be reduced using (27) leaving,

$$\frac{1}{2} \zeta_R'(-2\nu - 1, 1/2) + \zeta_R'(-2\nu - 2, 1/2) + \sum_{l=1}^{2\nu+2} \sum_{k=0}^{l} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!} \zeta_R'(-k, 1/2). \quad (28)$$

Adding the contribution (14) from (13), cancels the second term in (28) while that, (11), from (9) cancels the first one. These cancellations suggest that there is a yet more efficient way of organising the asymptotic limits.

The order of the summations in the remaining term in (28) is reversed by writing

$$\sum_{l=1}^{2\nu+2} \sum_{k=0}^{l} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!} \zeta_R'(-k, 1/2) = \sum_{k=0}^{2\nu+2} N_k^{(2\nu+1)} \zeta_R'(-k, 1/2)$$

where the vector of coefficients is

$$N_k^{(2\nu+1)} \equiv \sum_{l=k}^{2\nu+2} D_l^{(2\nu+1)} \frac{T^{(k)}(0, l)}{l!}$$

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with $D_0^{(2\nu+1)} = 0$.

Including the contribution (18) yields the final expression for (4),

$$A_\nu = \sum_{k=0}^{2\nu+2} N_k^{(2\nu+1)} \zeta_R'(-k, 1/2) + \int_0^1 t^{2\nu+1} T_{2\nu+2}''(t)\, dt. \tag{29}$$

The complete determinant for any odd ball is obtained by compounding (29) with the degeneracy expressed as a polynomial in p. For the 3-ball the degeneracy is $2p$ and the answer is

$$\zeta_3'(0) = -\frac{3}{32} + \zeta_R'(-2, 1/2) - \zeta_R'(-1, 1/2) + \frac{1}{4}\zeta_R'(0, 1/2) \approx -0.21139 \tag{30}$$

in agreement, after reverting to ordinary Riemann ζ–functions, with our earlier calculation [3] and with that of Bordag $et\ al$ [4], who use another direct method.

For the 5-ball the degeneracy is $p^3/3 - p/12$ and

$$\begin{aligned}\zeta_5'(0) = {} & \frac{47}{9216} + \frac{1}{12}\zeta_R'(-4, 1/2) - \frac{1}{6}\zeta_R'(-3, 1/2) + \frac{1}{24}\zeta_R'(-2, 1/2) \\ & + \frac{1}{24}\zeta_R'(-1, 1/2) - \frac{1}{64}\zeta_R'(0, 1/2) \approx 0.01375,\end{aligned} \tag{31}$$

which again agrees with [4] after rearrangement.

For the general d-ball it is straightforward to write a symbolic manipulation programme that performs all the operations automatically. Number enthusiasts might like to know that the 7-ball value is

$$\begin{aligned}\zeta_7'(0) = {} & -\frac{11831502329}{19263179980800} + \frac{1}{360}\zeta_R'(-6, 1/2) - \frac{1}{120}\zeta_R'(-5, 1/2) \\ & -\frac{1}{288}\zeta_R'(-4, 1/2) + \frac{1}{48}\zeta_R'(-3, 1/2) - \frac{41}{5760}\zeta_R'(-2, 1/2) \\ & -\frac{3}{640}\zeta_R'(-1, 1/2) + \frac{1}{512}\zeta_R'(0, 1/2) \approx -0.001751.\end{aligned} \tag{32}$$

5. Robin boundary conditions.

The essential equation in the Robin case is (cf [1])

$$\begin{aligned}A_\nu(\beta) = {} & \ln\Delta\big(-m^2\big) \\ \sim {} & \sum_p^* p^{2\nu+1}\left(p\ln\frac{2p}{p+\epsilon} + \epsilon - p + \frac{1}{2}\ln\frac{\epsilon}{p} - \ln(1+\beta/p)\right) \\ & + \sum_{p=1}^\infty p^{2\nu+1}\Bigg[\sum_{n=1}^{2\nu+2}\frac{R_n(\beta, t) - R_n(\beta, 1)}{\epsilon^n} + \sum_{n=1}^{2\nu+2} R_n(\beta, 1)\left(\frac{1}{\epsilon^n} - \frac{1}{p^n}\right) \\ & + \int_0^\infty\left(\frac{1}{2} - \frac{1}{\tau} - \sum_{k=1}^{\nu+1}(-1)^{k+1}B_{2k}\frac{\tau^{2k-1}}{(2k)!} + \frac{1}{e^\tau - 1}\right)\frac{e^{-\tau p}}{\tau}\, d\tau\Bigg]\end{aligned}$$

$$\tag{33}$$

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where the $R_n(\beta, t)$ are polynomials arising from a cumulant expansion of Olver's asymptotic series of the appropriate Robin combination of Bessel functions.

Most of the analysis is the same as for Dirichet conditions. We only note that

$$\sum_{p=1/2}^{*} p^{2\nu+1} \ln(1+\beta/p) = \frac{\gamma}{2(\nu+1)}\beta^{2\nu+2} + \int_0^\beta \beta^{2\nu+1}\big(2\psi(1+2\beta)-\psi(1+\beta)\big)\, d\beta$$
$$= \frac{\gamma+2\ln 2}{2(\nu+1)}\beta^{2\nu+2} + \int_0^\beta \beta^{2\nu+1}\psi(1/2+\beta)\, d\beta.$$
$$(34)$$

It is necessary, however, to be careful when working out the contribution of the term

$$\sum_{n=1}^{2\nu+2} R_n(\beta, 1) \sum_p^{*} p^{2\nu+1}\left(\frac{1}{\epsilon^n} - \frac{1}{p^n}\right) \tag{35}$$

because $R_n(\beta,1) = -(-1)^n\beta^n/n$ for even n and there are extra mass-independent terms when $\nu > 0$ coming from the application of (10) and (7).

Combining the different contributions produces the final expression for (33),

$$A_\nu(\beta) = \zeta_R'(-2\nu-1) + \sum_{k=0}^{2\nu+2} N_k^{(2\nu+1)}\, \zeta_R'(-k, 1/2)$$
$$+ \frac{\beta^{2\nu+2}}{4(\nu+1)}\sum_1^\nu \frac{1}{k} - \sum_{L=0}^{\nu-1}\beta^{2\nu-2L}\frac{\left(2^{-2L-1}-1\right)B_{2+2L}}{4(\nu-L)(L+1)} \tag{36}$$
$$- \int_0^\beta \beta^{2\nu+1}\psi(1/2+\beta)\, d\beta + \int_0^1 t^{2\nu+1}R_{2\nu+2}''(\beta, t)\, dt.$$

Specifically, for the 3-ball,

$$\zeta_3'(0) = \frac{3}{32}+\zeta_R'(-2, 1/2) + \frac{1}{2}\zeta_R'(-1, 1/2) + \frac{1}{4}\zeta_R'(0, 1/2) + \frac{\beta}{2}$$
$$- \int_0^\beta \beta\psi(1/2+\beta)\, d\beta, \tag{37}$$

and for the 5-ball,

$$\zeta_5'(0) = -\frac{61}{46080} + \frac{1}{12}\zeta_R'(-4, 1/2) + \frac{1}{6}\zeta_R'(-3, 1/2) + \frac{1}{24}\zeta_R'(-2, 1/2)$$
$$- \frac{1}{24}\zeta_R'(-1, 1/2) - \frac{1}{64}\zeta_R'(0, 1/2) + \frac{1}{24}\beta^4 + \frac{1}{24}\beta^3 + \frac{1}{48}\beta^2 - \frac{11}{576}\beta \quad (38)$$
$$+ \frac{1}{12}\int_0^\beta \beta(1-4\beta^2)\psi(1/2+\beta)\, d\beta,$$

which are again ultimately in agreement with [4].

The fact that the coefficients of the ζ–function derivatives are the same as the Dirichlet ones, up to signs, is a consequence of the special value $N_{2\nu+1}^{(2\nu+1)} = -1/2$.

6. An identity

Instead of using (7) directly to evaluate the mass-independent extra terms that come from (35), it is possible to expand $p^{2\nu} = (\epsilon^2 - p^2)^\nu$ binomially and use (7) for $\nu = 0$ on each term. This is equivalent to the known identity

$$B(\nu - L, L + 1) = \sum_{q=0}^{L} \frac{(-1)^L}{\nu - q} \binom{L}{q}$$

for the Euler β-function.

7. Comments

Although the odd ball results have been emphasised, the method applies equally well to the even case. One notes the oscillating variation of the determinants with dimension.

It is relatively easy to extend the results to the spinor Laplacian. One can also treat certain portions of the ball.

Acknowledgment

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